Lecture Oct 20

Some physical quantities associated to multiple integrals.

1. Mass

Let Ω be a solid in $\mathbb{R}^3$ with density function $\delta(x,y,z)$. A density function is a continuous function satisfying $\delta \geq 0$ in Ω.

The mass of the solid is defined to be

$$M = \iiint_{\Omega} \delta(x,y,z)\, dV(x,y,z).$$

For $D \subset \mathbb{R}^2$, the mass is

$$M = \iint_{D} \delta(x,y)\, dA(x,y)$$

D may be regarded as a thin object lying on the plane.

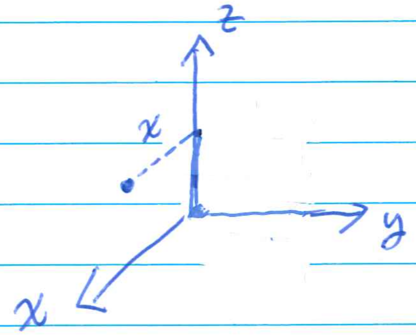
2. First moments.

Let P be a plane in $\mathbb{R}^3$. The first moment of Ω with respect to P is

$$M_P = \iiint_{\Omega} r(x,y,z)\, \delta(x,y,z)\, dV(x,y,z), \text{ where}$$

r is the signed distance to P.

When P is the yz-plane, signed distanc is $r(x,y,z) = x$



So,

$$M_{yz} = \iiint_{\Omega} x\, \delta(x,y,z)\, dV(x,y,z).$$

When P is the xz-plane,

$$M_{xz} = \iiint_{\Omega} y\, \delta(x,y,z)\, dV(x,y,z),$$

When P is the xy-plane,

$$M_{xy} = \iiint_{\Omega} z\, \delta(x,y,z)\, dV(x,y,z).$$

In the 2-dim case, the first moment with respect to a line (an axis) L is

$$M_L = \iint_{D} r(x,y,z)\, \delta(x,y)\, dA(x,y).$$

When L is the x-axis,

$$M_x = \iint_{D} y\, \delta(x,y)\, dA(x,y),$$

When L is the y-axis,

$$M_y = \iint_D x\,\delta(x,y)\,dA(x,y).$$

3. Center of Mass.

Let Ω be a solid with density δ in $\mathbb{R}^3$. The center of mass of Ω is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M}\iiint_\Omega x\,\delta(x,y,z)\,dV(x,y,z)$$

$$= \frac{M_{yz}}{M},$$

$$\bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

Call it the centroid when δ is constant.

Note that

$$0 = \frac{1}{M}\iiint_\Omega (x-\bar{x})\,\delta(x,y,z)\,dV(x,y,z)$$

$$0 = \frac{1}{M}\iiint_\Omega (y-\bar{y})\,\delta(x,y,z)\,dV(x,y,z)$$

$$0 = \frac{1}{M}\iiint_\Omega (z-\bar{z})\,\delta(x,y,z)\,dV(x,y,z).$$

That's, if we translate the origin of the coordinates to the center of mass, the center of mass of Ω in the new coordinates becomes $(0,0,0)$. In physics, setting the center of mass to be the origin simplifies some calculations.

When $D \subset \mathbb{R}^2$, the center of mass is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

e.g. 1 see e.g. 1 in 15.6

e.g. 2 see e.g 2 in 15.6

4. A symmetry result. $\Omega \subseteq \mathbb{R}^3$ is called symmetric w.r.t. the xy-plane if

$$(x,y,z) \in \Omega \Rightarrow (x,y,-z) \in \Omega.$$

A function f is odd in z if

$$f(x,y,-z) = -f(x,y,z).$$

Theorem Let f be a continuous fcn in Ω (symmetric w.r.t. xy-plane) which is odd in z. Then

$$\iiint_{\Omega} f(x,y,z)\, dV(x,y,z) = 0.$$

Proof. Consider the special case

$$\Omega = \{ (x,y,z) : -g(x,y) \leq z \leq g(x,y) \} \quad (x,y) \in D$$

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega^+} f\,dV + \iiint_{\Omega^-} f\,dV,$$

when $\Omega^+ = \Omega \cap \{z \geq 0\}$, $\Omega^- = \Omega \cap \{z \leq 0\}$.

$$\iiint_{\Omega^-} f(x,y,z)\,dV(x,y,z) = \iint_D \int_{-g(x,y)}^{0} f(x,y,z)\,dz\,dA(x,y)$$

$$= \iint_D \int_{g(x,y)}^{0} f(x,y,-t)(-dt)\,dA(x,y) \qquad (t=-z)$$

$$= \iint_D \int_{g(x,y)}^{0} -f(x,y,t)(-dt)\,dA(x,y)$$

$$= \iint_D \int_{g(x,y)}^{0} f(x,y,t)\,dt\,dA(x,y)$$

$$= -\iint_D \int_{0}^{g(x,y)} f(x,y,z)\,dz\,dA(x,y)$$

$$= -\iiint_{\Omega^+} f\,dV.$$

$$\therefore \iiint_{\Omega} f\,dV = \iiint_{\Omega^+} f\,dV - \iiint_{\Omega^+} f\,dV = 0. \ \#$$

In general, extend f to $\tilde{f}$ over $\mathbb{R}^3$ and let

R be a rectangular box $[a,b]\times[c,d]\times[-e,e]$ containing

Ω. One verifies that $\tilde{f}(x,y,-z) = -\tilde{f}(x,y,z)$ and

observe that

$$R = \{(x,y,z) : -e \le z \le e,\ (x,y) \in [a,b]\times[c,d]\}$$

that is $g(x,y) \equiv e$. So

$$\iiint_\Omega f\,dV \overset{\text{def}}{=} \iiint_R \tilde{f}\,dV = 0 \quad \text{(using the first step)}.$$

Corollary Let Ω be a solid symmetric w.r.t. xy-plane and δ is a constant. then $\bar{z} = 0$.

Pf: $\bar{z} = \frac{1}{M} M_{xy}$, $M_{xy} = \iiint_\Omega z\,dV$

here $f(x,y,z) = z$ is odd w.r.t. z, so $M_{xy} = 0$

and $\bar{z} = 0$.

You may formulate the 2-dim analogs.

5. Moments of Inertia

A solid rotating around an axis L in constant angular speed ω. Its kinetic energy is

$$\frac{1}{2} I_L \omega^2$$

when

$$I_L = \iiint_\Omega r^2 \delta \, dV, \quad \text{where}$$

$r(x,y,z)$ is the distance from $(x,y,z) \in \Omega$ to L. I_L is called the moment of inertia w.r.t. L. See 15.6 for further explanation.

When L is the x-axis, $r = \sqrt{y^2+z^2}$,

$$I_x = \iiint_\Omega (y^2+z^2) \, \delta \, dV$$

When L is the y-axis,

$$I_y = \iiint_\Omega (x^2+z^2) \, \delta \, dV,$$

When L is the z-axis

$$I_z = \iiint_\Omega (x^2+y^2) \, \delta \, dV.$$

For $D \subset \mathbb{R}^2$,

$$I_x = \iint y^2 \delta \, dA$$

$$I_y = \iint x^2 \delta \, dA$$

And the polar moment is

$$I_0 = I_x + I_y .$$

e.g. 3, e.g. 3 see 15.6.